Math 53, Discussion 105

Wednesday, April 22, 2020 9:31 AM

16.6 (parametric surfaces and their areas)
16.7 (surface integrals)

$$\int \underbrace{F\cdot\vec n}_{=f}\, dS \leftarrow 16.7$$

$$dS = \|r_u \times r_v\|\, dA \leftarrow 16.6$$

$$\vec n = \frac{r_u \times r_v}{\|r_u \times r_v\|}$$

$$F\cdot\vec n\, dS = F \cdot \frac{r_u\times r_v}{\cancel{\|r_u\times r_v\|}} \cdot \cancel{\|r_u\times r_v\|}\, dx\, dy$$

We can parametrize a surface as $r(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

The tangent plane of a surface at $r(u_0, v_0)$ is:

$$\left(r_u \times r_v\Big|_{(u_0,v_0)}\right)\cdot\left(\vec x - r(u_0,v_0)\right) = 0$$

The surface area of our surface is written as:

$$A(S) = \iint_D \|r_u \times r_v\|\, dA$$

D ← domain of r.

We can integrate a scalar function $f(x,y,z)$ over a surface S:

$$\iint_S f(x,y,z)\, dS = \boxed{\iint_D f(r(u,v))|r_u \times r_v|\, dA}$$

We can integrate a vector field over a surface by:

$$\iint_S F\cdot d\vec S = \iint_S F\cdot \vec n\, dS$$

$\vec n$ = unit normal

$$d\vec S = \vec n\, dS = \frac{r_u\times r_v}{\|r_u\times r_v\|}\, \|r_u \times r_v\|\, dA$$

Example: the sphere of R

$$r(u,v) = \langle R\sin u \cos(v), R\sin v \sin u, R\cos u\rangle$$

$0 \le u \le \pi$

$0 \le v \le 2\pi$

$$r_u = R\,\langle \cos(u)\cos(v), \cos(u)\sin(v), -\sin(u)\rangle$$

$$r_v = R\,\langle -R\sin(u)\sin(v), R\cos(v)\sin(u), 0\rangle$$

$$r_u \times r_v = R^2 \begin{pmatrix} \cos(u)\cos(v) \\ \cos(u)\sin(v) \\ -\sin(u)\end{pmatrix} \times \begin{pmatrix} -\sin(u)\sin(v) \\ \cos(v)\sin(u) \\ 0 \end{pmatrix}$$

$$= R^2 \begin{pmatrix} \cos(v)\sin^2(u) \\ \sin^2(u)\sin(v) \\ \underbrace{\cos^2(v)\cos(u)\sin(u) + \sin^2(v)\cos(u)\sin(u)}_{\cos(u)\sin(u)} \end{pmatrix}$$

$$\|r_u\times r_v\| = R^2\sqrt{\cos^2(v)\sin^4 u + \sin^4 u \sin^2 u + \cos^2(u)\sin^2(u)}$$

$$= \dots = R^2\sin(u)$$

$$A(S) = \int_0^\pi \int_0^{2\pi} R^2 \sin(u)\, dv\, du$$

$$= 2\pi R^2 \int_0^\pi \sin(u)\, du = 2\pi R^2(2) = 4\pi R^2$$

$dS =$ ...

$$= \frac{r_u \times r_v}{\|r_u \times r_v\|} \|r_u \times r_v\| \, dA$$

$= 4\pi R^2$

Exercises

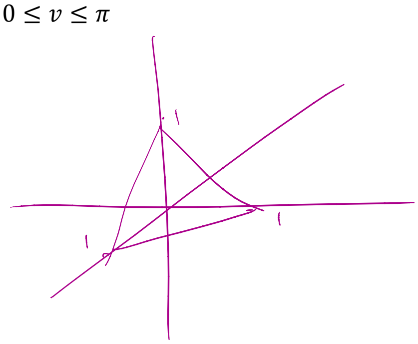
1)Find the equation of the tangent plane of the surface at the given point: $x = u^2, y = v^2, z = uv, u = 1, v = 1$

2) Find the area of the surface $z = \frac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$ for $0 \le x \le 1, 0 \le y \le 1$

$$\iint_D 1 \cdot \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA$$

3) Compute $\iint_S yz \, dS$ with S the plane $x + y + z = 1$ in the first octant

4) Compute $\int_S F \cdot dS$ for $F = \langle y, x, z^2 \rangle$ and S the heliocoid $\langle u \cos v, u \sin v, v \rangle$ $0 \le u \le 1, 0 \le v \le \pi$



1) $r(u,v) = \langle u^2, v^2, uv \rangle$

find the normal at $u=1$ $v=1$

$r_u \times r_v \Big|_{\substack{u=1 \\ v=1}}$ $\quad r_u = \langle 2u, 0, v \rangle$

$r_v = \langle 0, 2v, u \rangle$

$$r_u \times r_v = \begin{pmatrix} 2u \\ 0 \\ v \end{pmatrix} \times \begin{pmatrix} 0 \\ 2v \\ u \end{pmatrix} = \begin{bmatrix} -2v^2 \\ -2u^2 \\ 4uv \end{bmatrix}$$

$$r_u \times r_v \Big|_{\substack{u=1 \\ v=1}} = \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} \leftarrow \text{normal}$$

$r(1,1) = (1, 1, 1) \leftarrow$ point

$$\boxed{\langle -2, -2, 4 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 0}$$

$$\iint_S F \cdot dS = \iint_S F \cdot (r_u \times r_v) \, dA$$

$$r_u \times r_v = \begin{pmatrix} \cos(v) \\ \sin(v) \\ 0 \end{pmatrix} \times \begin{bmatrix} -u\sin(v) \\ u\cos(v) \\ 1 \end{bmatrix} = \begin{bmatrix} \sin(v) \\ -\cos(v) \\ u \end{bmatrix}$$

$$F \cdot (r_u \times r_v) = y\sin(v) - x\cos(v) + z^2 u$$

$$= \underbrace{u\sin^2(v) - u\cos^2(v)}_{-u\cos(2v)} + v^2 u$$